A new Petrov-Galerkin smoothed aggregation preconditioner for nonsymmetric linear systems

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Introduction We propose a new variant of smoothed aggregation (SA) suitable for nonsymmetric linear systems. SA is a highly successful and popular algebraic multigrid method for symmetric positive-definite systems [3, 2]. A relatively large number of significant parallel smoothed aggregation codes have been developed at universities, companies, and laboratories. Many of these codes are quite sophisticated and represent a significant investment in time and effort. Despite the large body of work on multigrid methods for fluid dynamics and the significant successes of smoothed aggregation, there have been surprisingly few attempts at generalizing the smoothed aggregation idea to nonsymmetric systems. Most smoothed aggregation variants for nonsymmetric systems either sacrifice performance on diffusion-dominated problems or do not perform well on highly convective problems. In this talk, a new variant is proposed that performs well in both the highly diffusive and highly convective regimes. The new algorithm is based on two key generalizations of SA: restriction smoothing and local damping. Restriction smoothing refers to the smoothing of a tentative restriction operator via a damped Jacobi-like iteration. Restriction smoothing is analogous to prolongator smoothing in standard SA and in fact has the same form as the transpose of prolongator smoothing when the matrix is symmetric. Local damping refers to damping parameters used in the Jacobi-like iteration. In standard SA, a single damping parameter is computed via an eigenvalue computation. Here, local damping parameters are computed by considering the minimization of an energy-like quantity for each individual grid transfer basis function. Restrictor Smoothing and Local Damping Let $A$ refer to a discretized partial differential equation, $P$ be an interpolation operator, and $R$ be a restriction operator. The key to any algebraic multigrid scheme is the precise definition of $P$ and $R$. In standard smoothed aggregation, $P$ is normally defined by

$$P = (I - \omega \text{diag}(A)^{-1}A)P^{(\text{tent})}$$

where $P^{(\text{tent})}$ is a simple easy to construct grid transfer that perfectly interpolates the near null space of $A$. This near null space corresponds to the true null space of $\tilde{A}$ where $\tilde{A}$ is identical to $A$ except that the boundary conditions are essentially ignored. The basic idea is that a simple prolongator is first developed which perfectly interpolates the lowest frequency mode. This prolongator is then improved by applying a Jacobi-like iteration to the prolongator basis functions. This effectively lowers basis function energy (measured in the $A$-norm) while maintaining the perfect interpolation of the lowest frequency. Restriction is then normally defined by taking $R = P^T$. In this talk, we consider a nonsymmetric generalization of smoothed aggregation. Unfortunately, generalization of the energy minimization concept is not at all obvious for nonsymmetric systems. What form should restriction and prolongator improvement take? Given that $A$ no longer defines a norm, what should be minimized? In the symmetric

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case, damping parameters are determined by minimizing eigenvalues. In the nonsymmetric case, should eigenvalues, singular values, or the field-of-values be considered? Additionally, should the absolute value be minimized or the largest real part? Yet another difficulty arises for problems with strong convective and strong diffusive regions. Should the damping parameter be chosen for the convective region or the diffusive region? In this talk, we propose two basic changes. The first corresponds to replacing $\omega$ in (1) with a diagonal matrix. This effectively replaces a single damping parameter over the entire domain with local damping parameters. The second change is that the restriction improvement step now takes the form:

$$R = (P^{(tent)})^T(I - A \ diag(A)^{-1}\Omega^{(r)})$$  \hspace{1cm} (2)

where $\Omega^{(r)}$ is a diagonal matrix consisting of damping parameters. Notice that when $A = A^T$, the above restrictor is simply the transpose of the prolongator. However, it is fundamentally different from $P^T$ when $A$ is nonsymmetric. (2) is similar in spirit to ideas explored in [1]. In this talk, it follows naturally by considering a Schur complement of a transformed $2 \times 2$ linear system. Given the simple form of the prolongator and restrictor, the main obstacle is to define suitable damping parameters. Using the $2 \times 2$ linear system framework, a suitable minimization principle is defined. This leads to a simple easy-to-compute formula for damping parameters. This formula effectively corresponds to minimizing the energy in the $A^T A$ norm of each individual grid transfer basis function. The main difficulty is that the minimization of individual basis functions gives a prolongator that no longer perfectly interpolates the lowest frequency mode. Given the importance of this property to standard smoothed aggregation method, we propose a relatively simple modification. This modification fixes the prolongator/restrictor obtained by the above local damping procedure so that the lowest frequency mode is properly addressed. While the resulting grid transfers do not have minimum energy, we will show that the basis functions have only slightly higher energy than those obtain before the modification. **Numerical Results** To evaluate the resulting multigrid algorithms, convergence results, sequential/parallel timings and multigrid operator complexities are reported. Several realistic compressible and incompressible flow examples as well as a semiconductor device modeling simulation are presented using both serial and parallel computing platforms. Our applications include discretizations arising from finite differences, finite elements, and finite volumes. In addition to demonstrating the overall effectiveness of the new scheme, the tests will be used to validate assumptions made in the development of the damping parameters. This include measures of the variation in local damping parameters as well as the increase in energy of prolongator basis functions associated with the perfect interpolation of the near null space.

**References**
